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Boundary element solutions of the steady state, singular, inverse heat transfer equation

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Abstract—A numerical investigation of the steady state, inverse heat conduction problem which is improperly posed and has a boundary singularity has been investigated. Using finite difference and finite element methods it is difficult to even mathematically model this problem. Therefore in this paper a minimal energy technique, which has been combined with a modified boundary element method, has been employed. The results obtained using this technique are convergent and stable and a comparison of the numerical solutions with analytical solutions, where available, are very encouraging.

1. INTRODUCTION

It is with great pleasure that we have this opportunity to officially celebrate the 70th birthday of Professor J. P. Hartnett. Professor Hartnett has throughout his long and distinguished research career been at the forefront of many novel approaches in heat transfer. Although the area of research which is presented in this paper has not been investigated by Professor Hartnett, we believe that the mathematical approach is in the spirit of much of his work and has a great potential for application in one of his present research interests, namely in viscoelastic fluids.

The steady state heat conduction problem for the temperature with constant physical properties, say u, satisfies the Laplace equation and if either u or $\partial u/\partial n$ is specified at all points on the boundary of a region (u must be specified on at least one point on the boundary), then u can be uniquely determined at all interior points of the region. This class of problems is well posed and can be solved using either the Finite Difference Method (FDM), Finite Element Method (FEM) or Boundary Element Method (BEM). However, in numerous experimental situations it is not always possible to specify a boundary condition at all points on the boundary of the physical domain where the solution is required. For example, impediments may arise in the measuring of the boundary data due to the boundary being unsuitable for attaching a sensor, or the accuracy of a boundary measurement may be seriously impaired by the presence of the sensor.

It is frequently possible to determine, or specify, either the function u or $\partial u/\partial n$ (i.e. the temperature or the heat flux) on only part of the boundary of the region and with no information available on the remaining part of the boundary. Clearly this is insufficient information in order to determine the function u everywhere within the solution domain. Fortunately, in many practical heat transfer applications extra sensors may be inserted into the interior region of interest and the temperature measured at these locations in order to provide more information. The question then arises as to whether given u, or $\partial u/\partial n$, on part of the boundary and u at a number of interior points of the domain, it is possible to determine uniquely the temperature distribution within the solution domain. Ingham *et al.* [1] introduced a minimal energy technique to the BEM and this has been successfully used to solve the Laplace equation with insufficient boundary information supplied in a rectangular domain. This minimal energy technique has been successfully extended to determine an unknown temperature-dependent thermal conductivity, ref. [2], and to solve the backward, unsteady heat conduction problem, ref. [3]. However, in all these problems no singularities on the boundary of the solution domain were present. In this paper we extend this minimal energy technique to deal with the solution of the steady state, heat transfer problem with constant coefficients in which the solution domain contains a boundary singularity.

Standard numerical methods, such as the FDM, the FEM and the BEM, for dealing with the boundary-value problems such as the Laplace equation tend, when using iterative methods, to suffer from having a slow rate of convergence as the mesh size decreases in the neighbourhood of boundary singularities, ref. [4]. Consequently, the possibility of modifying the standard techniques in order to give special treatment to the singular points and thereby to obtain solutions which converge more rapidly has received considerable attention. In particular for solving the Laplace equation in two dimensions Motz [5] and Woods [6] used a modification of the FDM, whilst Wait and Mitchell [7] used a modification of the FEM. In 1973 Symm [8] devised a Modified BEM to deal with the presence of singularities on the boundary of the solution domain for the two-dimensional Laplace equation, and Ingham and Kelmanson [9] extended this technique to solve the biharmonic equation.

In this paper we consider the problem in which there are insufficient boundary conditions prescribed for a unique solution of the Laplace equation to be

NOMENCLATURE

- E, G, EI, GI matrices for the Boundary Element Method
- $H(\Omega)$ Sobolev space
- $J(u)$ energy functional
- N number of boundary elements
- u temperature
- x, y dimensionless Cartesian coordinates.

Greek symbols

- α, β, γ, δ the first four coefficients of the expansion (2.13)
- Γ_0 set of interior points in Ω
- Γ_1 part of boundary on which no boundary condition is specified
- Γ_2 part of boundary on which boundary condition is given
- $\partial\Omega$ boundary of solution domain
- ϕ boundary value of Laplace equation
- ϕ' normal derivative of ϕ
- χ interior measurement data
- Ω solution domain.

obtained and on which there is a boundary singularity. The minimal energy method, combined with the Modified Boundary Element Method (MBEM), is used since it is extremely difficult even to pose a mathematical procedure to solve such problems when using the FDM and the FEM. In order to combine the minimal energy technique and the MBEM an iterative scheme has to be employed. The numerical procedure is illustrated by performing all of the calculations in an L-shape region, although the method may easily be extended to more complex geometries and where numerous singularities exist. The unknown temperature, u, is given on part of the boundary, say Γ_2, and further interior information has been given on a straight line, Γ_0 say, see Fig. 1. Extension of the work to more irregular-shaped boundaries and to the interior information being given at random positions within the solution domain is straightforward.

Let Ω be an L-shape domain in which the solution is sought, see Fig. 1, where O is the origin of the coordinate system and, for convenience, in order to illustrate the numerical technique we have taken

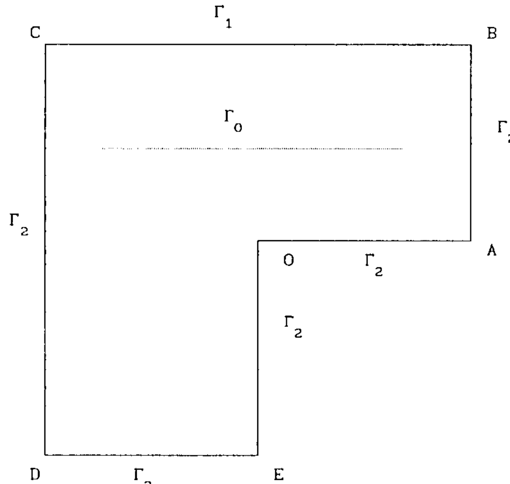

FIG. 1. The solution domain and the notation for problem (1.1)–(1.3).

OA = AB = 1 and BC = CD = 2. The governing equation for the temperature, i.e. the Laplace equation, in Ω is given by

$$\nabla^2 u = 0 \quad \text{in}\ \Omega \tag{1.1}$$

which has to be solved subject to the boundary condition

$$u = \phi \quad (\text{or}\ \partial u/\partial n = \phi') \quad \text{on}\ \Gamma_2 \tag{1.2}$$

and the interior measurement information

$$u(x,y) = \chi(x,y) \quad \text{on}\ \Gamma_0 \tag{1.3}$$

where ϕ, ϕ' and χ are given functions. Problem (1.1)–(1.3) is an example of an inverse heat conduction problem which is improperly posed and Hadamard [10] has pointed out that it is impossible to solve an improperly posed problem by the use of the classical theory of partial differential equations. Therefore a special treatment must be developed in order to solve the problem (1.1)–(1.3).

2. BOUNDARY ELEMENT METHOD

The fundamental basis of the BEM is Green's Integral Formula. For any sufficiently smooth function u, which satisfies the Laplace equation in $\Omega \subset \mathbb{R}^2$ with a piecewise smooth boundary $\partial\Omega$, we may write

$$\eta(\underline{p})\,u(\underline{p}) = \int_{\partial\Omega} u(\underline{q})\frac{\partial v(\underline{p},\underline{q})}{\partial n}\,\mathrm{d}s - \int_{\partial\Omega} \frac{\partial u(\underline{q})}{\partial n} v(\underline{p},\underline{q})\,\mathrm{d}s \tag{2.1}$$

where $v(\underline{p},\underline{q}) = \ln|\underline{p}-\underline{q}|$ is the fundamental solution of the Laplace equation, $\underline{p} \in \Omega \cup \partial\Omega$, $\underline{q} \in \partial\Omega$ and $\eta(\underline{p})$ is a constant which depends upon the location of the point $\underline{p}$ and is given by

$$\eta(\underline{p}) = \begin{cases} 2\pi & \text{when}\ \underline{p} \in \Omega \\ \theta & \text{when}\ \underline{p} \in \partial\Omega,\ \theta\ \text{is the angle between the tangents to}\ \partial\Omega\ \text{on either side of}\ \underline{p} \\ 0 & \text{when}\ \underline{p} \notin \Omega \cup \partial\Omega. \end{cases} \tag{2.2}$$

Taking the prime (′) to denote the derivative in the direction of the outward normal, the boundary value of $u(\underline{p})$ is $\phi(\underline{p})$ and the normal derivative is $\phi'(\underline{p})$ at $\underline{p}\in\partial\Omega$, then we have

$$\int_{\partial\Omega}\phi(\underline{q})\ln'|\underline{p}-\underline{q}|\,\mathrm{d}\underline{q}-\int_{\partial\Omega}\phi'(\underline{p})\ln|\underline{p}-\underline{q}|\,\mathrm{d}q = \eta(\underline{p})\,u(\underline{p}). \quad (2.3)$$

If we know either $\phi(\underline{p})$ or $\phi'(\underline{p})$ on $\partial\Omega$, we can obtain the other function by using equation (2.3). Then on substituting both these values into equation (2.1), we can obtain the solution of the steady state heat transfer equation everywhere in Ω.

Since the integral equations (2.1) and (2.3) can rarely be solved analytically we use a numerical scheme. The boundary $\partial\Omega$ is divided into N smooth intervals, $\partial\Omega_j$, where $j = 1,2,\ldots,N$, and on each segment ϕ and ϕ' are approximated by constant values ϕ_j and ϕ_j', where ϕ_j and ϕ_j' take the values of ϕ and ϕ' at the midpoints of the segment $\partial\Omega_j$, respectively. Therefore the integral formulae (2.1) and (2.3) become

$$\sum_{j=1}^{N}\phi_j\int_{\partial\Omega_j}\ln'|\underline{p}-\underline{q}|\,\mathrm{d}\underline{q}-\sum_{j=1}^{N}\phi_j'\int_{\partial\Omega_j}\ln|\underline{p}-\underline{q}|\,\mathrm{d}\underline{q} = \eta(\underline{p})\,u(\underline{p}) \quad (2.4)$$

$$\sum_{j=1}^{N}\phi_j\int_{\partial\Omega_j}\ln'|\underline{p}_i-\underline{q}|\,\mathrm{d}\underline{q}-\eta_i\phi_i - \sum_{j=1}^{N}\phi_j'\int_{\partial\Omega_j}\ln|\underline{p}_i-\underline{q}|\,\mathrm{d}\underline{q} = 0, \quad i = 1,2,\ldots,N \quad (2.5)$$

where $\underline{p}_i$ is the midpoint of the segment $\partial\Omega_i$ and $\eta_i = \eta(\underline{p}_i)$.

Using expresssion (2.5) we obtain

$$\sum_{j=1}^{N}\phi_j\left(\int_{\partial\Omega_j}\ln'|\underline{p}_i-\underline{q}|\,\mathrm{d}\underline{q}-\eta_i\delta_{ij}\right) - \sum_{j=1}^{N}\phi'\int_{\partial\Omega_j}(\ln|\underline{p}_i-\underline{q}|\,\mathrm{d}\underline{q}) = 0. \quad (2.6)$$

If we write

$$G_{ij} = \int_{\partial\Omega_j}\ln|\underline{p}_i-\underline{q}|\,\mathrm{d}\underline{q} \quad (2.7)$$

and

$$E_{ij} = \int_{\partial\Omega_j}\ln'|\underline{p}_i-\underline{q}|\,\mathrm{d}\underline{q}-\eta(\underline{p}_i)\,\delta_{ij} \quad (2.8)$$

where δ_{ij} is the Kronecker delta function, then equation (2.6) may be written as follows

$$\sum_{j=1}^{N}E_{ij}\phi_j-\sum_{j=1}^{N}G_{ij}\phi_j' = 0, \quad i = 1,2,\ldots,N. \quad (2.9)$$

The system of equations (2.9) contains N equations and $2N$ unknown variables, and if either ϕ_j or ϕ_j' is specified on each segment $\partial\Omega_j$, then the system of equations (2.9) may be solved and the other quantity may be determined.

Furthermore, if $\underline{p}_i$ $(i = 1,2,\ldots,k)$ are any set of interior points in the solution domain at which the solution is required then the integral equation (2.1) may be written in the form

$$2\pi u(\underline{p}_i) = \sum_{j=1}^{N}EI_{ij}\phi_j-\sum_{j=1}^{N}GI_{ij}\phi_j', \quad i = 1,\ldots,k \quad (2.10)$$

where

$$GI_{ij} = \int_{\partial\Omega_j}\ln|\underline{p}_i-\underline{q}|\,\mathrm{d}\underline{q} \quad (2.11)$$

and

$$EI_{ij} = \int_{\partial\Omega_j}\ln'|\underline{p}_i-\underline{q}|\,\mathrm{d}\underline{q}. \quad (2.12)$$

Thus given ϕ or ϕ' on $\partial\Omega$, the solution at any point $\underline{p}_i\in\Omega$ may be found directly by evaluating equation (2.10).

Using the standard BEM described above we can solve the steady state heat transfer equation in the domain Ω, if u or $\partial u/\partial n$ is specified on each segment on the boundary of the solution domain. However, this numerical technique tends to yield inaccurate solutions for problems which involve boundary singularities, for example the steady state heat transfer equation in an L-shape domain. Therefore, in order to obtain solutions which converge more rapidly a Modified Boundary Element Method (MBEM) was developed, see Symm [8] and Ingham and Kelmanson [9].

Let Ω be an L-shape domain, see Fig. 1, then in the neighbourhood of the re-entrant corner O the solution of the steady state heat transfer equation may be expressed as

$$w(\underline{p}) = \alpha+\beta r^{2/3}\cos(2\theta/3)+\gamma r^{4/3}\cos(4\theta/3) +\delta r^2\cos(2\theta)+\cdots \quad (2.13)$$

where α, β, ... are constants which are initially unknown and (r,θ) are polar coordinates centred at O. If we denote the first $2k$ terms of equation (2.13) for $w(\underline{p})$ by $w^*(\underline{p})$ and define

$$v(\underline{p}) = u(\underline{p})-w^*(\underline{p}) \quad (2.14)$$

then the new unknown function $v(\underline{p})$ satisfies the following problem

$$\begin{cases}\nabla^2 v = 0 \\ v|_{\partial\Omega} = u|_{\partial\Omega}-w^*|_{\partial\Omega} \quad \text{or} \\ \partial v/\partial n|_{\partial\Omega} = \partial u/\partial n|_{\partial\Omega}-\partial w^*/\partial n|_{\partial\Omega}.\end{cases} \quad (2.15)$$

Solving problem (2.15) using the standard BEM we obtain a system of n-linear equations with $n+2k$ unknowns. In order to balance the number of equations and unknowns we assume that both v and $\partial v/\partial n$ are given at the $2k$ segments near the re-entrant point O, and hence reduce the number of unknowns to n.

Symm [8] indicated that the results obtained were not satisfactory if $k = 1$ but accurate approximate solutions could be obtained by setting $k = 2$. Hence in this paper we assume that $k = 2$, i.e. w^* contains the first 4 terms in expansion (2.13) for w.

3. MATHEMATICAL MODEL

3.1. *Minimal energy method*

Ingham *et al.* [1] have shown that the direct and the least-squares methods were unsuitable for solving improperly posed inverse problems when using the BEM. Therefore, in this paper we have used a minimal energy technique to solve a slight modification of problem (1.1)–(1.3), i.e. we have investigated the solution of the related problem

$$\begin{cases} \nabla^2 u = 0 & \text{in } \Omega \\ u = \phi & \text{on } \Gamma_2 \\ |u|_{\Gamma_0} - \chi| \leqslant \varepsilon \end{cases} \tag{3.1}$$

where $\varepsilon > 0$ is a pre-assigned small quantity. Clearly the solution of problem (1.1)–(1.3) is one of the solutions of problem (3.1), and the problem now reduces to finding the solution of problem (3.1) which is continuously dependent on the boundary conditions and the measurement data.

We let $H^1(\Omega)$ denote the usual Sobolev space in the domain Ω, ref. [11], and

$$\overset{*}{H}{}^1(\Omega) = \{v \in H^1(\Omega);\ v = \phi \text{ almost everywhere on } \Gamma_2\}$$

$$\overset{*}{H}{}^{1/2}(\partial\Omega) = \{\text{the trace of } v \text{ on } \partial\Omega;\ v \in \overset{*}{H}{}^{1/2}(\Omega)\}$$

$$\overset{*}{H}{}^{1/2}(\Gamma_1) = \{\text{the restriction of } v \text{ on } \Gamma_1;\ v \in \overset{*}{H}{}^{1/2}(\partial\Omega)\}$$

for any $\psi \in \overset{*}{H}{}^{1/2}(\Gamma_1)$, we let

$$\phi^* = \begin{cases} \phi & \text{on } \Gamma_2 \\ \psi & \text{on } \Gamma_1 \end{cases} \tag{3.2}$$

then $\phi^* \in \overset{*}{H}{}^{1/2}(\partial\Omega)$. We now consider the following boundary value problem

$$\begin{cases} \nabla^2 u = 0 & \text{in } \Omega \\ u = \phi^* & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

We know that problem (3.3) has a unique weak solution such that $u(x) \in \overset{*}{H}{}^{1/2}(\Omega)$, ref. [12], and in the domain Ω the function $u(x)$ is a harmonic. Hence we can define an operator

$$\begin{cases} \mathbb{A} : \overset{*}{H}{}^{1/2}(\Gamma_1) \longrightarrow H^1(\Omega) \\ \mathbb{A}v = u(x) \qquad \forall v \in \overset{*}{H}{}^{1/2}(\Gamma_1) \end{cases} \tag{3.4}$$

where $u(x)$ is the weak solution of problem (3.3). If there is a $\psi \in \overset{*}{H}{}^{1/2}(\Gamma_1)$ such that $\mathbb{A}\psi = \chi$ or $\|\mathbb{A}\psi|_{\Gamma_0} - \chi\|_{H^1(\Gamma_0)}$ is sufficiently small and $\mathbb{A}\psi$ is continuously dependent on the data, then in order to obtain an approximate solution of problem (1.1) by the BEM we use this function ψ as the boundary condition Γ_1.

We write

$$a(u, v) = \iint_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}y \tag{3.5}$$

$$J(u) = \frac{1}{2} a(u, u) = \frac{1}{2} \iint_\Omega |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{\partial\Omega} u \frac{\partial u}{\partial n} \,\mathrm{d}s \tag{3.6}$$

where n is the outward normal to the boundary $\partial\Omega$. We know that $J(u)$ describes the thermal energy of the steady field, ref. [13], and hence $J(u)$ may be called the energy functional.

Now consider a subset of $\overset{*}{H}{}^{1/2}(\Gamma_1)$ such that

$$K = \{v; v \in \overset{*}{H}{}^{1/2}(\Gamma_1),\quad |\mathbb{A}v|_{\Gamma_0} - g| \leqslant \varepsilon\} \tag{3.7}$$

where $\varepsilon > 0$ is a small pre-assigned constant and from the definition of the operator $\mathbb{A}$ we obtain the subset of $H^1(\Omega)$, namely

$$S = \mathbb{A}K. \tag{3.8}$$

Clearly S is a closed convex set in $H^1(\Omega)$. If the solution of problem (3.1) exists then S is not empty. So problem (3.1) is equivalent to the variational problem

$$J(u) = \inf_{v \in S} J(v). \tag{3.9}$$

Han [14] has proved that there is a unique solution of equation (3.9), and the solution is smooth in Ω if the functions ϕ and g are sufficiently smooth. In view of the definition of the operator $\mathbb{A}$ we know that the variational problem (3.9) is now equivalent to

$$J(\mathbb{A}\bar{\psi}) = \inf_{\psi \in K} J(\mathbb{A}\psi) \tag{3.10}$$

which on discretization becomes

$$J(u) = \frac{1}{2} \sum_{i,j=1}^{N} \int_{\partial\Omega} \phi_i \,\partial\phi_j/\partial n \,\mathrm{d}s$$

$$= \frac{1}{2} \int_{\partial\Omega} \bar{\phi}^{\mathrm{T}} G^{-1} E \bar{\phi} \,\mathrm{d}s \tag{3.11}$$

where $\bar{\phi} = (\phi_1^*, \phi_2^*, \ldots, \phi_N^*)^{\mathrm{T}}$. We assume that $\bar{\psi} = (\psi_1, \ldots, \psi_M)^{\mathrm{T}}$ is an unknown variable on Γ_1 and the remaining $N - M$ elements of $\bar{\phi}$ are denoted by $\tilde{\phi}$. The constraint condition $\psi \in K$ may be written

$$\left| \sum_{j=1}^{N} \frac{1}{2\pi} (EI_{ij} - GI_{il} G_{lm} E_{mj}) \bar{\phi}_j - \chi_i \right| \leqslant \varepsilon,$$

$$i = 1, \ldots, \mathscr{k}. \tag{3.12}$$

Let W denote the M columns of the matrix $(EI - GIG^{-1}E)$ which is related to the unknown temperature on Γ_1, i.e. $\bar{\psi}$ and W_1 denote the remaining $N - M$ columns of the matrix $(EI - GIG^{-1}E)$, then expression (3.12) becomes

$$|W\bar{\psi} + W_1\tilde{\phi} - \chi| \leqslant \varepsilon \tag{3.13}$$

and if $\bar{\chi} = \chi - W_1\bar{\phi}$ then we have

$$|W\bar{\psi} - \bar{\chi}| \leqslant \varepsilon. \tag{3.14}$$

It is clear that

$$\min J(u) = \min\left(\int_{\partial\Omega} \bar{\phi}^{\mathrm{T}} G^{-1} E\bar{\phi}\, \mathrm{d}s\right)$$

is equivalent to $\min J(u) = \min(\bar{\phi}^{\mathrm{T}} G^{-1} E\bar{\phi})$ and the problem reduces to finding $\bar{\psi} = (\psi_1, \ldots, \psi_M)^{\mathrm{T}}$ which satisfies

$$\begin{cases} \bar{J}(\mathbb{A}\bar{\psi}) = \min \bar{\phi}^{\mathrm{T}} G^{-1} E\bar{\phi} \\ |W\bar{\psi} - \bar{\chi}| \leqslant \varepsilon. \end{cases} \tag{3.15}$$

Solving the constrained minimal problem (3.15) using the NAG routine E04UCF, we obtain the function $u(x)$ on Γ_1 and hence an approximate solution of problem (1.1)–(1.3) using the standard BEM may be obtained. The NAG routine E04UCF is designed to minimize an arbitrary smooth function subject to certain constraints which may include simple bounds on the variables, linear constraints and nonlinear constraints and the method is a sequential quadratic programming method, ref. [15].

3.2. *Numerical scheme*

The mathematical model described in Section 3.1 has been successfully employed to solve some inverse heat conduction problems [1–3]. However, all the problems considered contain no singularities on the boundary of the solution domain. As mentioned in Section 2, the boundary singularity leads to inaccurate solutions when using the standard BEM. Therefore a MBEM has to be employed in order to improve the accuracy of the numerical solution. Since the unknowns, $\alpha, \beta, \ldots$, which are introduced in the MBEM, do not appear in the minimization equation (3.15), then it is not possible to minimize the problem using the minimal energy technique as described in Section 3.1. Hence we devised the following iterative scheme:

Step 1. Specify the boundary condition, say ϕ, on Γ_2 and the interior information, say χ, on Γ_0.

Step 2. Guess the values of the coefficients α, β, γ and δ. In all the examples presented in this paper we have taken $\alpha = \beta = \gamma = \delta = 0$ as the guessed values.

Step 3. Let $v(p) = u(p) - w^*(p)$, where w^* contains the first 4 terms in the expansion (2.13) for w. We now consider the inverse problem

$$\begin{aligned} &\nabla^2 v = 0 \\ &v|_{\Gamma_2} = u|_{\Gamma_2} - w^*|_{\Gamma_2} \\ &v|_{\Gamma_0} = u|_{\Gamma_0} - w^*|_{\Gamma_0}. \end{aligned} \tag{3.16}$$

On solving problems (3.16), using the technique as described in Section 3.1, we obtain the next approximation to the values of u on Γ_1, say ψ.

Step 4. Using the MBEM solve the following boundary value problem

$$\begin{aligned} &\nabla^2 u = 0 \\ &u|_{\Gamma_2} = \phi|_{\Gamma_2} \\ &u|_{\Gamma_1} = \psi|_{\Gamma_1} \end{aligned} \tag{3.17}$$

and thus we obtain the next approximation to the values of the constants α, β, γ and δ.

Step 5. The iteration is considered to have converged when both the difference between two successive iterative values of ψ and the coefficients in the MBEM, are sufficiently small, i.e. when

$$\max_{\Gamma_1} |\psi_m - \psi_{m-1}| < \varepsilon_1 \tag{3.18}$$

and

$$|\alpha_m - \alpha_{m-1}| + |\beta_m - \beta_{m-1}| + |\gamma_m - \gamma_{m-1}| + |\delta_m - \delta_{m-1}| < \varepsilon_2 \tag{3.19}$$

where ε_1 and ε_2 are two pre-assigned small values and the subscript m denotes the number of iterations. If both expressions (3.18) and (3.19) are satisfied then the process is complete and the unknown boundary values and the solution everywhere in the solution domain determined. Otherwise return to Step 3.

We note that in Step 2 the first guess for the values of the constants α, β, γ and δ have been set to be zero. However, numerous calculations have been performed with other values for these constants and it has always been found that the choice of the values for these constants does not have any significant effect on the number of iterations required for convergence or on the accuracy of the final results. Because w is a local solution of the steady state heat transfer equation in the neighbourhood of the re-entrant corner O, then the value of u on Γ_1 can only generate a small error in the coefficients of w^*. Therefore, no large errors are generated when a poor guess of the values of the constants α, β, γ and δ has been chosen.

4. NUMERICAL RESULTS

The choice of the values of the two control parameters ε_1 and ε_2 has been thoroughly investigated. It has been found that in all the examples considered in this paper, $\varepsilon_1 = 10^{-3}$ and $\varepsilon_2 = 10^{-5}$ are sufficiently small so that any further decrease in the value of these parameters does not produce any changes in the results.

4.1. *Example* 1

Here we take a very simple function, $u(x, y) = x^2 - y^2$, as the test function and impose the following boundary conditions:

Table 1. The values of α, β, γ, δ and $J(u)$ for each successive iteration for Example 1 and with $N = 80$

Number of iterations	α	β	γ	δ	$J(u)$
1	0.00247	−0.0018	−0.0008	0.99843	4.02268
2	0.00097	−0.0005	−0.00028	0.99929	4.01927
3	0.00018	−0.0001	−0.00004	0.99981	4.01728
4	0.00002	-10^{-5}	-10^{-6}	0.99992	4.01594
5	10^{-6}	-10^{-7}	-10^{-7}	0.99998	4.01436
6	10^{-7}	-10^{-7}	-10^{-8}	0.99998	4.01337

$$\phi(x,y) = \begin{cases} x^2, & 0 \leqslant x \leqslant 1, y = 0 \\ 1-y^2, & x = 1, 0 \leqslant y \leqslant 1 \\ 1-y^2, & x = -1, -1 \leqslant y \leqslant 1 \\ x^2-1, & -1 \leqslant x \leqslant 0, y = -1 \\ -y^2, & x = 0, -1 \leqslant y \leqslant 0 \end{cases} \tag{4.1}$$

and interior data information

$$\chi(x) = x^2 - 0.25, \quad x \in \Gamma_0 \tag{4.2}$$

where $\Gamma_0 = \{-0.75 \leqslant x \leqslant 0.75, \ y = y_0\}$ and $y_0 = 0.5$. Clearly, the solution is such that $\alpha = \beta = \gamma = 0$ and $\delta = 2$ and numerical solutions have been obtained with $N = 40$, 80 and 160.

Table 1 shows how the coefficients α, β, γ and δ in expansion (2.13) for w^* and the value of the energy function $J(u)$ converge as the number of iterations increase when taking $N = 80$. On evaluating equation (3.15) analytically for the function $J(u)$ we obtain $J(u) = 4$. We observe that after 5 iterations the error in the numerical solution for the values of the constants α, β, γ and δ are within about 0.01% and for the value of $J(u)$ is about 0.2%. Further, when using as the first guess for the values of the constants α, β, γ and δ the value 1, then a very similar accuracy and number of iterations are obtained. This result indicates that the first guess for the values of constants α, β, γ and δ does not significantly affect the accuracy of the numerical approximate solution.

Figure 2 illustrates the analytical and numerical solutions for the value of the temperature $u(x, y)$ on Γ_1 as the value of N varies. It is observed that the agreement between the present numerical solution and the analytical solution is excellent. Further, the numerical solution appears to be converging to the analytical solution as the number of discretizations increase.

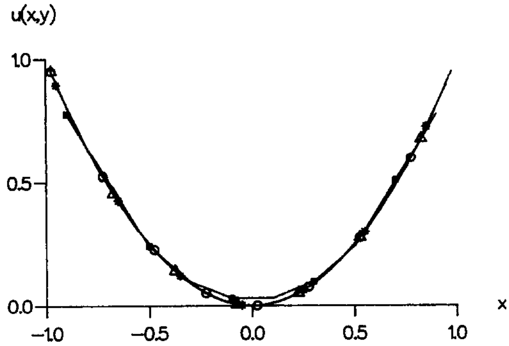

FIG. 2. The values of the temperature $u(x, y)$ on Γ_1 for Example 1, where ○ is the analytical solution, ■ is the numerical solution with $N = 40$, * with $N = 80$, and △ with $N = 160$.

We have also investigated the effect of the position of the set of interior measurement points Γ_0 on the accuracy of the numerical procedure. For example, we have taken $y_0 = 0.1$, 0.3, 0.5, 0.7 and 0.9, and in order to illustrate the accuracy of the technique we present results only for the case when $N = 80$. The corresponding values of the energy function $J(u)$ are 4.0138, 4.0139, 4.0134, 4.0133 and 4.0132, respectively. This result indicates that when using the minimal energy method, the location of the interior measurement information is not very important and the function $J(u)$ may be predicted to within about 0.2%.

4.2. *Example* 2

Here we consider a mixed boundary value problem for which there is no simple analytical solution by taking

$$\partial u(x,0)/\partial n = 0, \qquad 0 \leqslant x \leqslant 1, y = 0 \tag{4.3}$$

$$u(x,1) = 0, \qquad x = 1, 0 \leqslant y \leqslant 1 \tag{4.4}$$

$$u(-1,y) = 1, \qquad x = -1, -1 \leqslant y \leqslant 1 \tag{4.5}$$

$$\partial u(x,-1)/\partial n = 0, \qquad -1 \leqslant x \leqslant 1, y = -1 \tag{4.6}$$

$$\partial u(0,y)/\partial n = 0, \qquad x = 0, -1 \leqslant y \leqslant 0. \tag{4.7}$$

In order to test the accuracy of the numerical technique developed in this paper we assume that $\chi(x, y)$ on Γ_0 is evaluated using $\partial u(x,y)/\partial n|_{\Gamma_1} = 0$ and the solution is obtained using the MBEM as described by Symm [8].

Table 2 shows how the coefficients in expression (2.13) for w^* and the value of $J(u)$ converge as the number of iterations increase for the case $N = 80$. It is observed that the numerical solution again appears to be convergent and stable even though there is no simple analytical solution to this problem. Further, the lines of constant $u(x, y)$ using the present numerical technique with $N = 40$, 80 and 160, respectively, give results which are graphically indistinguishable.

Table 3 shows the numerical solutions which have been obtained for the temperature $u(x, y)$ on Γ_1 with

Table 2. The values of α, β, γ, δ and $J(u)$ for each successive iteration for Example 2 and with $N = 80$

Number of iterations	α	β	γ	δ	$J(u)$
1	0.66769	−0.45398	−0.21637	0.00218	0.58322
2	0.66723	−0.45337	−0.21588	0.00084	0.58183
3	0.66704	−0.45294	−0.21535	0.00038	0.57958
4	0.66688	−0.45258	−0.21513	0.00016	0.57896
5	0.66676	−0.45232	−0.21498	0.00009	0.57839
6	0.66669	−0.45211	−0.21490	0.00004	0.57794
7	0.66666	−0.45206	−0.21487	0.00002	0.57762
8	0.66666	−0.45205	−0.21487	0.00002	0.57743

(i) w^* being set to be zero, (ii) w^* containing the first two terms in expansion (2.13) for w, and (iii) w^* containing the first four terms in expansion (2.13) for w. In order to check the accuracy of the present numerical technique the numerical solution as obtained when using the MBEM with $\partial u/\partial n = 0$ given on Γ_1 is also included in Table 3. The results indicate that the solution obtained using the standard BEM with the minimal energy method leads to relatively large errors, typically up to 10%. The MBEM improved the accuracy of the approximate solution, but the relative error when only the first two terms are used in the MBEM is still about 4%. Therefore, in order to obtain an accurate approximate solution it is necessary to take the first four terms in the MBEM and then results which are accurate to 0.3% everywhere in the solution domain may be obtained.

Sections 4.1 and 4.2 illustrate that the numerical solutions of problem (1.1) which are obtained when using the present numerical technique are convergent as the number of discretizations increase and the solution is independent of the location of the interior measurement point. However, since this heat conduction problem with insufficient boundary data is an improperly posed problem then it is necessary to consider the stability of the numerical technique. In order to do this we again consider Example 2, but with the addition of a small perturbation to the measurement data, $\chi(x, y)$, and investigate the size of the errors which are generated by this small perturbation.

4.3. *Example* 3

In this example the same boundary conditions as those used in Example 2 are used but a small perturbation $\delta(x, y)$ is added to the function $\chi(x, y)$, i.e.

$$\chi^*(x, y) = \chi(x, y) + \delta(x, y), \quad (x, y) \in \Gamma_0 \quad (4.8)$$

where $|\delta(x, y)| \leqslant \varepsilon_3$ and ε_3 is a pre-assigned small quantity and $\delta(x, y)$ is given stochastically. Solutions have been obtained for numerous values of N but in order to illustrate the accuracy of the results we present results only for $N = 80$.

Table 4 illustrates a typical set of results for the temperature $u(x, y)$ on Γ_1 for $\varepsilon_3 = 0$, 10^{-2} and 10^{-4}. It is observed that only very small errors are generated when a relatively small perturbation is input on the interior measurement data. Clearly, since the perturbation function is given stochastically, the results presented are only typical results. However, numerous calculations have been performed and in all cases the results were always found to be in error by the same order of magnitude as those presented in Table 4.

We have also investigated the effect of introducing a small perturbation onto the boundary conditions. Consider Example 2 again, but replace the boundary conditions (4.4) and (4.5) by

Table 3. The numerical solution for the temperature $u(x, y)$ on Γ_1 for Example 2

x	Solution of problem (1.1)–(1.3)			Solution with
	$w^* = 0$	2 terms	4 terms	u given on Γ_1
−0.9	1.0658	1.0062	0.95502	0.95327
−0.7	0.93623	0.89834	0.86975	0.86892
−0.5	0.85982	0.81722	0.78264	0.78175
−0.3	0.75471	0.72167	0.69183	0.69023
−0.1	0.63245	0.61593	0.59356	0.59344
0.1	0.45638	0.47115	0.49102	0.49134
0.3	0.33569	0.36438	0.38319	0.38470
0.5	0.22193	0.25246	0.27338	0.27473
0.7	0.10348	0.13618	0.16173	0.16267
0.9	0.00629	0.02021	0.04916	0.04945

Table 4. A typical numerical solution for the temperature $u(x, y)$ on Γ_1 for various values of ε_3 and x for Example 3 and with $N = 80$

x	Numerical solution		
	$\varepsilon_3 = 10^{-2}$	$\varepsilon_3 = 10^{-4}$	$\varepsilon_3 = 0$
−0.9	0.95583	0.95508	0.95502
−0.7	0.87008	0.86979	0.86975
−0.5	0.78223	0.78268	0.78264
−0.3	0.69204	0.69185	0.69183
−0.1	0.59367	0.59359	0.59356
0.1	0.39078	0.49100	0.49102
0.3	0.38296	0.38315	0.38319
0.5	0.27319	0.27332	0.27338
0.7	0.16158	0.16169	0.16173
0.9	0.04902	0.04911	0.04916

Table 5. A typical variation of $J(u)$ for various values of the perturbation bounds and N for Example 3

	$J(u)$				
N	$\varepsilon_4 = 0.0$ $\varepsilon_5 = 0.0$	0.01 0.01	0.0001 0.01	0.01 0.0001	0.0001 0.0001
40	0.57797	0.5885	0.5837	0.5833	0.5796
80	0.57743	0.5833	0.5811	0.5809	0.5772
160	0.57728	0.5812	0.5795	0.5793	0.5753

$$u(x,1) = 0+\delta_1(y), \quad x = 1, \quad 0 \leqslant y \leqslant 1 \tag{4.9}$$

$$u(-1,y) = 1+\delta_2(y), \quad x = -1, \quad -1 \leqslant y \leqslant 1 \tag{4.10}$$

where $|\delta_1(y)| \leqslant \varepsilon_4$ and $|\delta_2(y)| \leqslant \varepsilon_5$, ε_4 and ε_5 are pre-assigned small quantities and $\delta_1(y)$ and $\delta_2(y)$ are given stochastically.

Again we find that the lines of constant $u(x,y)$ in the solution domain as obtained without perturbation, see Example 2, are indistinguishable when ε_4 and ε_5 are sufficiently small, say 10^{-2}. Table 5 illustrates a typical result for the variation of the energy function $J(u)$ for various values of ε_4 and ε_5 and N. It is again observed that the numerical technique is convergent and stable when a small perturbation is added to the boundary conditions since a small perturbation generates only a small error in the numerical solution.

5. CONCLUSIONS

In this paper we have illustrated, by means of examples, the use of the MBEM in the solution to an improperly posed steady state, inverse heat transfer problem in which there is a boundary singularity. The minimal energy method with an iterative scheme has been developed and it has been found that the numerical solution is convergent and stable as the number of discretizations increase, and where analytical solutions exist the numerical solution is in excellent agreement. We have also investigated the effect of introducing a small perturbation into measurement data and the boundary conditions and we have found that the numerical technique is always stable.

It is important to observe that when the minimization problem is solved numerically, a good starting guess is not important and in all the examples presented in this paper the starting guesses for all the variables were set to be zero. Further, since the minimization problem is in a positive definite quadratic form, there is a unique minimum solution and therefore a poor starting guess will not result in another local minimum solution being obtained.

Although only an L-shape domain has been considered in the numerical examples, no difficulties have been experienced when extending this technique to more complicated solution domain geometries. Further, any number of boundary singularities may be treated using the MBEM technique combined with the minimal energy method.

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